34. Antiderivative

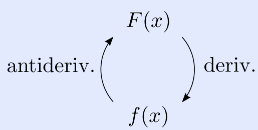
34.1. Introduction

Instead of starting with a function and asking what its derivative is, we turn things around in this section:

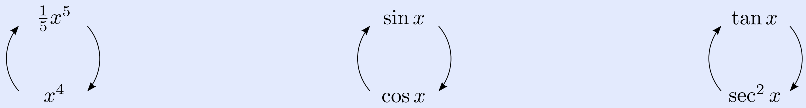
- Find a function F that has derivative x^4.

Since we know that the derivative of a power of x reduces the power by one, we take as an initial guess $F(x) = x^5$. Check: $F'(x) = 5x^4$. Because of the factor of 5, this is not quite what we wanted, but we now know how to adjust. Let $F(x) = \frac{1}{5}x^5$. Check: $F'(x) = x^4$ (yes).

If F and f are functions and $F'(x) = f(x)$, then F is called an **antiderivative** of f. For instance, $F(x) = \frac{1}{5}x^5$ is an antiderivative of $f(x) = x^4$. The relationship between derivatives and antiderivatives can be represented schematically:

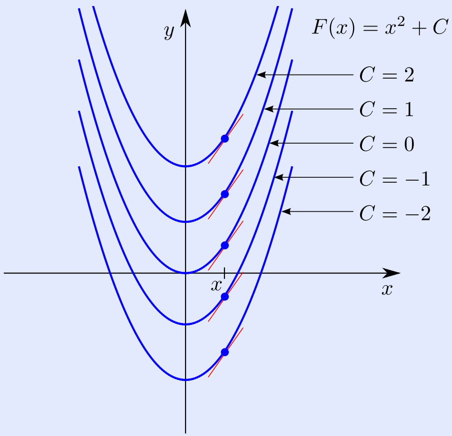


For instance,



34.2. Indefinite integral

Let $f(x) = 2x$. The function $F(x) = x^2$ is an antiderivative of f. But so is $F(x) = x^2 + 1$, and $F(x) = x^2 + 2$. In fact, $F(x) = x^2 + C$ is an antiderivative of f for *any* constant C. The graph of $F(x) = x^2 + C$ is the graph of $F(x) = x^2$ shifted vertically by C units, so we have the following picture:



Each function pictured is an antiderivative of $2x$, that is, each function has the same derivative (= general slope function) $2x$. This agrees with the observation that at any x the tangents to the graphs are all parallel, which implies that their slopes are the same.

A question remains: Are the functions $F(x) = x^2 + C$ (C any real number) *all* of the possible antiderivatives of $f(x) = 2x$? Since the graph of any antiderivative has to share with the graphs drawn above the stated property of parallel tangent lines, it is reasonable to expect that there can be no further antiderivatives. This is in fact, the case. The the main step in the verification is the following result, which says roughly that the only way a function can have a derivative that is constantly 0 is if it is a constant function (so that its graph is a horizontal line).

Function with zero derivative is constant. If $f'(x) = 0$, then $f(x) = C$ for some constant C.

The theorem says that if f has general slope function 0 (that is, every tangent is horizontal), then f must be a constant function (that is, its graph must be a horizontal line), and this seems reasonable.

Here is the careful verification: Assume that $f'(x) = 0$. Suppose $f(x)$ is not constant. Then $f(a) \neq f(b)$ for some $a \neq b$. By the mean value theorem, there exists c between a and b such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

But the expression on the right is nonzero since $f(a) \neq f(b)$ so we have $f'(c) \neq 0$, in violation of our assumption that $f'(x) = 0$ for all x. We conclude that $f(x)$ is constant, that is, $f(x) = C$ for some constant C.

Two antiderivatives differ by constant. If F is any antiderivative of f, then every antiderivative of f is of the form $F(x) + C$ for some constant C.

Here is the verification: Let F be an antiderivative of f and suppose that G is another antiderivative of f. We have $F'(x) = f(x)$ and also $G'(x) = f(x)$, so that $F'(x) = G'(x)$. Let $H(x) = G(x) - F(x)$. Then $H'(x) = G'(x) - F'(x) = 0$. By the previous theorem (with H playing the role of f), $H(x) = C$ for some constant C. Then $G(x) = F(x) + H(x) = F(x) + C$, as claimed.

The result just verified establishes the earlier claim that the functions $F(x) = x^2 + C$ (C any real number) are the only antiderivatives of $f(x) = 2x$.

Indefinite integral. If f is a function and F is any antiderivative of f, we write

$$\int f(x)\,dx = F(x) + C \quad (C\text{, arbitrary constant})$$

and call it the **(indefinite) integral** of f.

For example, since x^2 is an antiderivative of $2x$, we have

$$\int 2x\,dx = x^2 + C.$$

Saying that C is an “arbitrary” constant, is saying that it can be any real number. So in a sense,

$$\int 2x\,dx$$

simultaneously represents

$$x^2 + 0, \quad x^2 + 1, \quad x^2 + \tfrac{1}{2}, \quad \ldots, \quad x^2 + \text{ any number}$$

and these are precisely all of the possible antiderivatives of $2x$ (according to the previous theorem). For this reason, the indefinite integral of f is often called the most general antiderivative of f.

The reason for the notation $\int f(x)\,dx$ will be given later, but for now it can be regarded as a Leibniz notation for the most general antiderivative of f. The function $f(x)$ between the symbols $\int$ and dx is called the **integrand**. If an independent variable other than x is used, then dx is changed accordingly. For instance, we would write $\int t^4\,dt = \frac{1}{5}t^5 + C$.

34.3. Integral rules

Any derivative rule gives rise to an integral rule (and conversely). For example,

$$\begin{aligned}
\frac{d}{dx}[\sin x] = \cos x \quad &\Rightarrow \quad \int \cos x\,dx = \sin x + C \\
\frac{d}{dx}[\tan x] = \sec^2 x \quad &\Rightarrow \quad \int \sec^2 x\,dx = \tan x + C \\
\frac{d}{dx}[e^x] = e^x \quad &\Rightarrow \quad \int e^x\,dx = e^x + C \\
\frac{d}{dx}[x^n] = nx^{n-1} \quad &\Rightarrow \quad \int nx^{n-1}\,dx = x^n + C
\end{aligned}$$

The last integral rule is not very convenient; we would prefer to have a rule for the integral of simply x^n. Such a rule follows:

Power rule for integrals.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad (n \neq -1).$$

In words, the integral of a power of x equals x to the one higher power over that higher power, plus an arbitrary constant. For example,

$$\int x^4\,dx = \frac{x^5}{5} + C,$$

in agreement with what we found earlier using trial and error. The power rule is verified (as is any integral rule) by checking the validity of the corresponding derivative rule:

$$\frac{d}{dx}\left[\frac{x^{n+1}}{n+1}\right] = \frac{d}{dx}\left[\tfrac{1}{n+1}x^{n+1}\right] = x^n,$$

as desired.

The power rule excludes the case $n = -1$ (as it must since this would produce a zero in the denominator). The omitted case is neatly handled by an earlier derivative rule:

Integral of 1/x.

$$\int x^{-1}\,dx = \ln|x| + C$$

This formula holds since $\dfrac{d}{dx}[\ln|x|] = \dfrac{1}{x} = x^{-1}$.

Constant multiple rule. For any constant c,

$$\int cf(x)\,dx = c\int f(x)\,dx$$

SUM/DIFFERENCE RULE.

$$\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$$

34.3.1 Example Find $\int \left(5x^3 + 7x^2\right)\,dx$.

Solution Using the sum/difference rule, the constant multiple rule, and the power rule, we get

$$\begin{aligned}\int \left(5x^3 + 7x^2\right)\,dx &= \int 5x^3\,dx + \int 7x^2\,dx \\ &= 5\int x^3\,dx + 7\int x^2\,dx \\ &= 5\left(\frac{x^4}{4} + C_1\right) + 7\left(\frac{x^3}{3} + C_2\right),\end{aligned}$$

where we have used subscripts on the arbitrary constants since one cannot assume that the constants are equal. After distributing and collecting terms, we get

$$\int \left(5x^3 + 7x^2\right)\,dx = 5\left(\frac{x^4}{4}\right) + 7\left(\frac{x^3}{3}\right) + (5C_1 + 7C_2) = \frac{5x^4}{4} + \frac{7x^3}{3} + C,$$

where $C = 5C_1 + 7C_2$. As C_1 and C_2 range through all real numbers, C ranges through all real numbers as well. Therefore, we can forget about C_1 and C_2 and just regard C as an arbitrary constant. In short, one can apply the sum rule, the difference rule, and the constant multiple rule, ignoring any arbitrary constants, provided a single arbitrary constant is appended at the end. □

Now that we have seen in detail how the rules work, we can suppress steps.

34.3.2 Example Find $\int \left(7+4x^3-\frac{5}{x^2}+\frac{6}{x}+2\sqrt[3]{x^2}\right)\,dx$.

Solution We do some rewriting in order to use the power rule:

$$\begin{aligned}
&\int \left(7+4x^3-\frac{5}{x^2}+\frac{6}{x}+2\sqrt[3]{x^2}\right)\,dx \\
&\quad= \int \left(7x^0+4x^3-5x^{-2}+6x^{-1}+2x^{2/3}\right)\,dx \\
&\quad= 7\left(\frac{x^1}{1}\right)+4\left(\frac{x^4}{4}\right)-5\left(\frac{x^{-1}}{-1}\right)+6\ln|x|+2\left(\frac{x^{5/3}}{5/3}\right)+C \\
&\quad= 7x+x^4+\frac{5}{x}+6\ln|x|+\frac{6\sqrt[3]{x^5}}{5}+C.
\end{aligned}$$

□

Verifying a stated integral formula is different from finding an integral. To verify an integral formula, it is only necessary to verify the corresponding derivative formula:

34.3.3 Example Verify that

$$\int \sqrt{1-x^2}\,dx = \tfrac{1}{2}\left(x\sqrt{1-x^2}+\sin^{-1}x\right)+C.$$

Solution We verify the corresponding derivative formula:

$$\begin{aligned}\frac{d}{dx}&\left[\tfrac{1}{2}\left(x\sqrt{1-x^2}\ +\ \sin^{-1}x\right)\right]\\ &= \frac{1}{2}\left(1\cdot\sqrt{1-x^2}+x\cdot\tfrac{1}{2}\left(1-x^2\right)^{-1/2}(-2x)+\frac{1}{\sqrt{1-x^2}}\right)\\ &= \frac{1}{2}\left(\sqrt{1-x^2}-\frac{x^2}{\sqrt{1-x^2}}+\frac{1}{\sqrt{1-x^2}}\right)\\ &= \frac{1}{2}\left(\frac{2-2x^2}{\sqrt{1-x^2}}\right)\\ &= \sqrt{1-x^2}.\end{aligned}$$

□

34.3.4 Example Find $\displaystyle\int\left(4e^x+\sin x-\frac{8}{x}+\frac{3}{1+x^2}\right)dx$.

Solution We use the sum/difference rule and the constant multiple rule, and then the fact pointed out above that every derivative rule gives rise to a corresponding integral rule (rewriting the second term in order to use the rule):

$$\begin{aligned}\int&\left(4e^x+\sin x-\frac{8}{x}+\frac{3}{1+x^2}\right)dx\\ &= 4\int e^x\,dx-\int(-\sin x)\,dx-8\int\frac{1}{x}\,dx+3\int\frac{1}{1+x^2}\,dx\\ &= 4e^x-\cos x-8\ln|x|+3\tan^{-1}x+C.\end{aligned}$$

(See 25 for a list of derivative rules.) □

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34.4. Initial value problem

If f is an unknown function and we know only that it has derivative $f'(x) = 2x$, then we cannot hope to determine f precisely since there are infinitely many functions with derivative $2x$ (namely, $x^2 + C$ for any constant C). However, if we also know the value of f for some particular input, for instance $f(2) = 5$, then we can determine f precisely:

34.4.1 Example Find the function f given that $f'(x) = 2x$ and $f(2) = 5$.

Solution The equation $f'(x) = 2x$ says that the desired function f is an antiderivative of $2x$, so

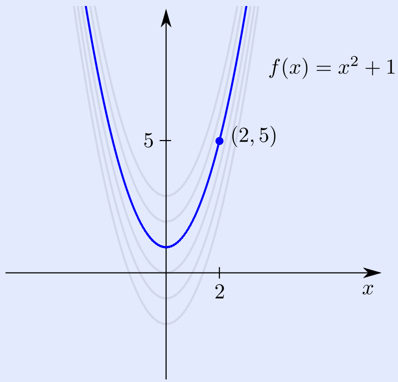
$$f(x) = \int 2x\,dx = 2\left(\frac{x^2}{2}\right) + C = x^2 + C,$$

that is, $f(x) = x^2 + C$. Using the condition $f(2) = 5$, we get

$$5 \underset{\substack{\uparrow \\ \text{given}}}{=} f(2) \underset{\substack{\uparrow \\ \text{evaluate}}}{=} 2^2 + C.$$

The preceding equations reveal that $5 = 4 + C$, so that $C = 1$. Therefore, $f(x) = x^2 + 1$.

Knowledge of the general slope function $f'(x) = 2x$ of the desired function f allowed us to determine that $f(x) = x^2 + C$. At that point we could tell that the graph of f had the shape of the parabola $y = x^2$, but there was an unknown vertical shift of C units that kept us from completely determining f. The second condition, $f(2) = 5$, said that the graph of f had to go through the point $(2, 5)$, and from this we determined that the shift amount had to be 1:



□

The problem of finding a function given its derivative and its value for some particular input is called an **initial value problem**. Many modeling problems in the sciences and engineering are initial value problems.

34.4.2 Example A car traveling northeast on I-85 (assumed to be straight) has velocity at time t hr given by $v(t) = 20t + 55$ mph. Given that the car is at Lagrange after one hour, find where the car began its trip.

Solution Let $f(t)$ be the car's position, relative to Lagrange, at time t. Since velocity is the rate at which position changes, we have $v(t) = f'(t)$, which says that f is an antiderivative of v, so

$$f(t) = \int v(t)\,dt = \int (20t + 55)\,dt = 20\left(\frac{t^2}{2}\right) + 55t + C = 10t^2 + 55t + C,$$

that is, $f(t) = 10t^2 + 55t + C$. Since the car is at Lagrange after one hour, its position at time $t = 1$ is 0, that is, $f(1) = 0$. Therefore,

$$0 \underset{\substack{\uparrow \\ \text{given}}}{=} f(1) \underset{\substack{\uparrow \\ \text{evaluate}}}{=} 10(1)^2 + 55(1) + C.$$

The preceding equations reveal that $0 = 65 + C$, so that $C = -65$. Therefore, $f(t) = 10t^2 + 55t - 65$. The car began its trip at time $t = 0$, so its initial position was $f(0) = -65$. In other words, the car was 65 miles southwest of Lagrange (and therefore around Tuskegee).

The information $v(t) = 20t + 55$ tells us the car's speedometer reading at any time t. From this alone, we could not have hoped to determine the car's initial position (one can imagine cars at various places along I-85 always having identical speedometer readings). It was the additional information that the car was at Lagrange after one hour that allowed for the determination of its initial position (and, in fact, its position at any time).

In terms of the graph, the speedometer readings allowed us to determine that the position function was $f(t) = 10t^2 + 55t + C$, so we could tell that the graph had the shape of the upturning parabola $y = 10t^2 + 55t$ but with an unknown vertical shift of C units. The additional information told us that the graph had to go through the point $(1, 0)$ and this allowed for the determination of the shift amount. □